Lecture 16 on Nov. 07 2013

In the last lecture, an index of z_0 with respect to a closed curve γ has been introduced. Now we study some properties of the index $n(\gamma, z_0)$. Since z_0 keeps away from γ, we can find a tiny disk $B(z_0, \epsilon)$ so that $B(z_0, \epsilon)$ has no intersection with γ. here $B(z_0, \epsilon)$ denotes the disk centered at z_0 with radius ϵ. Clearly if ϵ is small enough, we have $|z - w| \geq c^*$ for all z in $B(z_0, \epsilon)$ and w on γ. c^* is a positive constant. Choosing z an arbitrary point in $B(z_0, \epsilon)$, we have

$$\begin{aligned} |n(\gamma, z) - n(\gamma, z_0)| &= \frac{1}{2\pi} \left| \int_\gamma \frac{1}{w - z} - \frac{1}{w - z_0} \, \mathrm{d}w \right| = \frac{|z - z_0|}{2\pi} \left| \int_\gamma \frac{1}{(w - z)(w - z_0)} \, \mathrm{d}w \right| \\ &\leq \frac{|z - z_0|}{2\pi} \int_\gamma \frac{1}{|w - z||w - z_0|} \, |\mathrm{d}w|. \end{aligned}$$

By our assumption, we know that $|w - z| \geq c^*$ and $|w - z_0| \geq c^*$. Hence from the above estimate, we imply that

$$|n(\gamma, z) - n(\gamma, z_0)| \leq \frac{|z - z_0|}{2\pi (c^*)^2} \int_\gamma |\mathrm{d}w| = \frac{\text{length of } \gamma}{2\pi (c^*)^2} \, |z - z_0|.$$

Therefore if z is very close to z_0, equivalently if ϵ (the radius of $B(z_0, \epsilon)$) is very small, we have $|n(\gamma, z) - n(\gamma, z_0)| < 1/2$. In light that $n(\gamma, z)$ and $n(\gamma, z_0)$ are all integers, we show that $n(\gamma, z) = n(\gamma, z_0)$ must hold. In other words, all points in $B(z_0, \epsilon)$ share same index with respect to γ. Given z_1 and z_2 two points in $\mathbb{C}$ and a continuous path l connecting z_1 and z_2, if the intersection of l and γ is empty, then we can cover l by a finite sequence of tiny balls. Meanwhile all points in each tiny disk share same index. Supposing we have two tiny balls in the sequence say B_1 and B_2, then $B_1 \cap B_2 \neq \emptyset$. otherwise l is not continuous. all points in B_1 have same index, denoted by N_1 and all points in B_2 have same index denoted by N_2. But $B_1 \cap B_2 \neq \emptyset$. Therefore $N_1 = N_2$. In other words, if we can connect z_1 and z_2 by a continuous path l whose intersection with γ is empty, then z_1 and z_2 have same index.

Now we come back to the Cauchy integral formula. From the last lecture, we know that if $n(\gamma, z_0) \neq 0$, then

$$f(z_0) = \frac{1}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{w - z_0} \, \mathrm{d}w.$$

From the above arguments, choosing ϵ small enough, then $n(\gamma, z) = n(\gamma, z_0)$ for all z in $B(z_0, \epsilon)$. Hence by Cauchy integral formula, we know that

$$f(z) = \frac{1}{2\pi i n(\gamma, z)} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w = \frac{1}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w.$$

Moreover if s is small enough, we also have

$$f(z + s) = \frac{1}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{w - (z + s)} \, \mathrm{d}w.$$

Therefore it holds that

$$\frac{f(z + s) - f(z)}{s} = \frac{1}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{(w - z - s)(w - z)} \, \mathrm{d}w.$$

Taking $s \to 0$, the right-hand side above converges to

$$\frac{1}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{(w - z)^2} \, \mathrm{d}w.$$

Hence we know that f is derivable at z and it holds that

$$f'(z) = \frac{1}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{(w-z)^2} \, \mathrm{d}w.$$

Inductively the higher order derivatives of f can also be calculated. it is the proposition in the following

Proposition 0.1. *If f is analytic in Δ where Δ is a disk, then for any natural number k,*

$$f^{(k)}(z) = \frac{k!}{2\pi i n(\gamma, z_0)} \int_\gamma \frac{f(w)}{(w-z)^{k+1}} \, \mathrm{d}w.$$

Here γ is a contour in Δ such that $n(\gamma, z_0) \neq 0$. z is any point in $B(z_0, \epsilon)$ with ϵ small enough.

From Proposition 0.1, two cheap results can be easily obtained.

Theorem 0.2 (Liouville's theorem)**.** *If f is analytic on $\mathbb{C}$ and $|f(z)| \leq M$ for some $M > 0$ and all z in $\mathbb{C}$, then f must be a constant.*

Proof. Fixing z in $\mathbb{C}$ and R large enough so that z is in $B(0, R/2)$. Here $B(0, R/2)$ is the ball centered at 0 with radius $R/2$. Then by Proposition 0.1, we know that

$$|f'(z)| = \frac{1}{2\pi} \left| \int_{|z|=R} \frac{f(w)}{(w-z)^2} \, \mathrm{d}w \right|.$$

Here the abosolute value of the index in Cauchy formula is 1 in that the index of z with respect to the circle $|z| = R$ must be 1 or -1. Using the above equality, we show that

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|z|=R} \frac{|f(w)|}{|w-z|^2} \, |\mathrm{d}w|.$$

Noticing that z is in $B(0, R/2)$, so for all w on $|z| = R$, $|w - z| \geq R/2$. Applying this estimate together with the fact that $|f| \leq M$ to the above inequality, we know that

$$|f'(z)| \leq \frac{2M}{\pi} \frac{1}{R^2} \int_{|z|=R} |\mathrm{d}w| = \frac{2M}{\pi} \frac{1}{R^2} 2\pi R = \frac{4M}{R} \longrightarrow 0, \qquad \text{as } R \to \infty.$$

This shows that $f'(z) = 0$. Since z is arbitrary, therefore $f'(z) = 0$ for all z in $\mathbb{C}$ which tells us that f must be a constant. ☐

The second theorem is Morera's theorem. It gives us a way to go from continuity to analyticity.

Theorem 0.3 (Morera's theorem)**.** *if f is continuous in a domain Ω and for all γ a closed curve in Ω we have*

$$\int_\gamma f(z) \, \mathrm{d}z = 0,$$

then f must be analytic.

Proof. Using the condition in Theorem 0.3, we know that

$$F(z) = \int_{\gamma(z_0, z)} f(w) \, \mathrm{d}w$$

is well-defined. here z_0 is a fixed point in Ω, $\gamma(z_0, z)$ is a path in Ω connecting z_0 and z. From the previous arguments, we know that $F(z)$ is analytic and $f(z) = F'(z)$. Using Proposition 0.1, we know that F can be differentiated infinitely many times. So from the relationship $f(z) = F'(z)$, we know that f must also be differentiated infinitely many times. So f is analytic. ☐